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A Theorem in Nonserial Dynamic Programming*

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This note presents a new theorem in nonserial dynamic programming. This theorem, which generalizes a previous one of the authors, allows, in the cases in which it can be employed, cutting down the computational effort for solving the secondary optimization problem.

1. INTRODUCTION

This note presents a new result in nonserial dynamic programming. Its intelligibility demands the knowledge of the introductory section of one among References [1], [2], and [3].

2. THE MATHEMATICAL RESULT

DEFINITION. Consider an interaction graph $G(X, \Gamma)$. Let E be the set of the edges of G, $X' = \{x_1, x_2, ..., x_m\} \subset X$ $(m \geqslant 2)$, $x_i \in X'$ and $x_j \in X'$. The subset X' is *quasi fully connected* if, for every two vertices x_i and x_j nonconnected in G, E contains all the edges (x_i, x_k) and (x_j, x_k) for all $k \in (\{1, 2, ..., m\} - \{i, j\})$.

THEOREM. *Let G be an interaction graph with $D(G) \geqslant k$. Let $x_i \in X$, $\Gamma(x_i)$ be quasi fully connected and $|\Gamma(x_i)| = k$. Then there exists an order of elimination which begins with x_i and has minimal dimension.*

Proof. A minimal dimension order not beginning with x_i, if such order exists, begins either with a vertex $x_j \in \Gamma(x_i)$ or with one of the remaining vertices of X.

Consider the former case first. It is easy to see that, if $\Gamma(x_i)$ is quasi fully connected, the graph which results from the elimination of x_i is a subgraph

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of the one obtained eliminating x_j. Hence, since $D(G) \geqslant k$, it is clear that there exists a minimal dimension order beginning with x_i.

In the latter case let $y_1, y_2, \ldots, y_l$ be the vertices eliminated in the minimal dimension order before the elimination of a vertex in the set $\{x_i\} \cup \Gamma(x_i)$.

In the graph resulting from the elimination of $y_1, y_2, \ldots, y_l$ from G, since the set $\Gamma(x_i)$ is still quasi fully connected, for what was said above, there exists a minimal dimension order beginning with x_i.

Since $y_j \notin \Gamma(x_i)$ for $j \in \{1, 2, \ldots, l\}$, it is clear that the partial dimension (see [1] or [2]) of the two orders $x_i, y_1, y_2, \ldots, y_l$ and $y_1, y_2, \ldots, y_l, x_i$ are equal and that the graph resulting from the elimination of the variables $y_1, y_2, \ldots, y_l$, x_i does not depend upon the order of their elimination (see Theorem 4.1 of [3]). This completes the derivation. Q.E.D.

This theorem is a generalization of Theorem 4 of Reference [1]. In fact let $k = 2$ and let x_i be a vertex with degree two in G and $\Gamma(x_i) = \{x_j, x_k\}$. Then, in such case, the theorem states that, if the interaction graph has dimension greater or equal to two (i.e., if it is not a tree), since the subset $\{x_j, x_k\}$ is quasi fully connected whether x_j and x_k are or are *not* connected in G, then there exists a minimal dimension order beginning with x_i.

This is precisely the statement of Theorem 4 of reference [1]. An example in which the theorem stated above can be applied is found in Fig. 1.

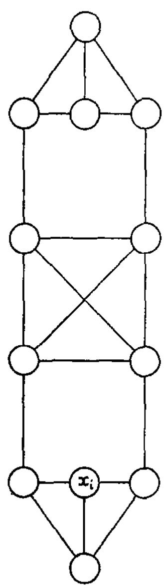

Fig. 1. An example of an interaction graph.

Since the interaction graph of Fig. 1 contains a fully connected subgraph with four vertices, its dimension is clearly three or more and hence there exists a minimal dimension order in which the vertex x_i, connected to a quasi fully connected subset of three vertices, is eliminated first.

It is easy to see that, in this case, the repetitive application of the theorem stated (and of Theorem 1 of [1]) allows finding a minimal dimension order without employing the algorithms of [1] and [3].

References

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